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Note on Singular Solutions.

By J. M. Page.

If, as is customary, we designate by the symbol

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

a transformation group of one parameter in two variables, and by

$$U'f \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} + \eta'(x, y, y')\frac{\partial f}{\partial y},$$

the *once-extended* group corresponding to Uf, the condition—as Lie has repeatedly shown—that an ordinary differential equation of the first order

$$\Omega(x, y, y') = 0 \tag{1}$$

shall be *invariant* under the group Uf, is that the expression

$$U'(\Omega) \equiv \xi\frac{\partial \Omega}{\partial x} + \eta\frac{\partial \Omega}{\partial y} + \eta'\frac{\partial \Omega}{\partial y'}$$

shall be zero, either identically, or by means of $\Omega = 0$. Further, it is well known that if Uf is not *trivial* with respect to (1), that is, if the path-curves of the group of one parameter, Uf, are not identical with the integral curves of the differential equation (1), this equation may be integrated by a quadrature.

A limited number of the path-curves of Uf may, however, coincide with particular integral curves of (1) when Uf is *not* trivial with respect to (1); and along these curves, if any such exist, the values of y' given by Uf, namely,

$$y' \equiv \frac{\eta(x, y)}{\xi(x, y)}, \tag{2}$$

must coincide with the values of y' given by (1).

To find such curves, therefore, as are at once path-curves of Uf and particular integral curves of the invariant differential equation (1), we only need to substitute the value of y' from (2) in (1), so that

$$\Omega\left(x, y, \frac{\eta}{\xi}\right) = 0 \tag{3}$$

will represent the required curves.

But it is readily seen that the envelope of the integral curves of (1), if one exists, must also be included in (3). For, since the family of integral curves of (1) is, as a whole, invariant under the group Uf, it is clear that the envelope must be an invariant curve whose points are interchanged by means of Uf, that is, the envelope must be a path-curve of the group Uf, and, at the same time, a curve whose equation satisfies (1).

Hence if the ordinary differential equation (1) possesses a *Singular Solution*, the singular solution will be included in equation (3). If (3) breaks up into factors, of course each factor must be separately examined to see whether it is a particular integral of (1) or a Singular Solution.

It may be remarked that it is impossible for ξ and η to be both *zero* along the envelope which represents the Singular Solution of (1); for, as we saw above, the points of the envelope must be interchanged when the curves of the invariant family are interchanged, whereas all points on the curves, along which $\xi \equiv \eta \equiv 0$, are absolutely invariant under the group Uf.

Since every differential equation of the first order in two variables admits of a group of one parameter Uf, and since we can usually give by inspection the groups of which the differential equations treated in the ordinary text-books admit, the above is a simple and expeditious method of finding Singular Solutions of ordinary differential equations of the first order, when such exist. The various superfluous loci—the cusp-loci, node-loci, etc.—which are also given by the ordinary method, are thus avoided.

For example, the equation

$$y'^3 - 4xyy' + 8y^2 = 0,$$

with the complete primitive

$$y = c\,(x - c)^2,$$

is invariant under the group

$$Uf \equiv x\,\frac{\partial f}{\partial x} + 3y\,\frac{\partial f}{\partial y}\,.$$

Hence, from (3),

$$\frac{27y^3}{x^3} - 4y^2 = 0,$$

or

$$y = 0, \; y = \frac{4x^3}{27}.$$

The first equation is seen to be a particular integral, while the second is a Singular Solution.

LEIPZIG, *November*, 1895.